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The vertex visibility number of graphs

Dhanya Roy [a,*] Gabriele Di Stefano [b,†] Sandi Klavžar [c,d,e,‡]
Aparna Lakshmanan S [a,§]

[a] Department of Mathematics, Cochin University of Science and Technology, Cochin - 22, Kerala, India

[b] Department of Information Engineering, Computer Science and Mathematics, University of L'Aquila, Italy

[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[e] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

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Abstract

If $x \in V(G)$, then $S \subseteq V(G) \setminus \{x\}$ is an x-visibility set if for any $y \in S$ there exists a shortest x,y-path avoiding S. The x-visibility number $v_x(G)$ is the maximum cardinality of an x-visibility set, and the maximum value of $v_x(G)$ among all vertices x of G is the vertex visibility number $\mathrm{vv}(G)$ of G. It is proved that $\mathrm{vv}(G)$ is equal to the largest possible number of leaves of a shortest-path tree of G. Deciding whether $v_x(G) \geq k$ holds for given G, a vertex $x \in V(G)$, and a positive integer k is NP-complete even for graphs of diameter 2. Several general sharp lower and upper bounds on the vertex visibility number are proved. The vertex visibility number of Cartesian products is also bounded from below and above, and the exact value of the vertex visibility number is determined for square grids, square prisms, and square toruses.

[*] dhanyaroyku@gmail.com, dhanyaroyku@cusat.ac.in
[†] gabriele.distefano@univaq.it
[‡] sandi.klavzar@fmf.uni-lj.si
[§] aparnaren@gmail.com, aparnals@cusat.ac.in

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1 Introduction

General position and mutual-visibility are very active areas of metric and algorithmic graph theory. The concepts are complementary to each other, and a progress in one of the areas typically has an impact on the other area. The general position problem has been recently surveyed in [5], see also [1, 16, 19, 23]. For the mutual-visibility problem we refer to the seminal paper [10] and the following selection of studies [3, 4, 7, 9, 13, 14, 15, 17, 19, 24].

In [22], the general position version was investigated from a vertex's point of view. Among the many motivations for this research, let us highlight the following. In [11, Chapter III] it is shown how to place a set of points with integer positive coordinates (i,j), $j \leq i$, such that each point is in mutual-visibility with $(0,0)$ and such that the number of points with the same abscissa is maximized. The problem turns out to be very interesting, as the solution has links with the Farey series and the Euler's totient function ϕ: the number of points with abscissa n is exactly $\phi(n)$.

Due to the above motivation (and more), the paper [22] gives the following definitions. If x is a vertex of a graph G, then $S \subseteq V(G)$ is said to be an *x-position set* if for any $y \in S$, no vertex of $S \setminus \{y\}$ lies on any shortest x,y-path. The *vertex position number* $\mathrm{vp}(G)$ of G is the maximum cardinality of an x-position set among all vertices x of G. The paper [22] yields numerous results dealing with the largest and smallest orders of maximum x-position sets, in particular giving bounds in terms of the girth, vertex degrees, diameter and radius.

In this paper, we complement the investigation from [22] by considering the mutual-visibility problem from a vertex's point of view. If x is a vertex of a graph G, then $S \subseteq V(G) \setminus \{x\}$ is an *x-visibility set* if for any $y \in S$ there exists a shortest x,y-path P, such that $V(P) \cap S = \{y\}$. The *x-visibility number* $v_x(G)$ is the maximum cardinality of an *x-visibility set*, we also say that $v_x(G)$ is the *visibility number of x*. An *x-visibility set* of cardinality $v_x(G)$ is called a *v_x-set*. The maximum value of $v_x(G)$ among all vertices x of G is called the *vertex visibility number* $\mathrm{vv}(G)$ of G, and a corresponding v_x-set is a vv-*set*. For instance, if $n \geq 3$, then $v_x(C_n) = 2$ for any $x \in V(C_n)$, hence $\mathrm{vv}(C_n) = 2$. Similarly, if $n \geq 3$, then $v_x(P_n) = \deg_{P_n}(x)$, hence $\mathrm{vv}(P_n) = 2$. It is worth mentioning the following fact.

Lemma 1.1 *If x is a leaf of a connected graph G with $n(G) \geq 3$, then $v_x(G) < \mathrm{vv}(G)$.*

The paper is organized as follows. In the following subsection, further definitions needed are listed. In Section 2 we first prove the fundamental reason for the computational difficulty of the vertex visibility number: $\mathrm{vv}(G)$ is equal to the largest possible number of leaves of a shortest-path tree of G. This builds a bridge between the new introduced concepts and the Dijkstra's algorithm, the result of which is precisely a tree of shortest-paths. In the second main result of the section we prove that the x-VISIBILITY problem is NP-complete even for graphs of diameter 2, where the x-VISIBILITY problem asks whether $v_x(G) \geq k$ holds for given G, a vertex $x \in V(G)$, and a positive integer k. In Section 3 we deduce some general sharp lower and upper bounds on the vertex visibility number in terms of the order, the maximum degree, and the eccentricity. The vertex visibility number of Cartesian products is investigated in Section 4. After establishing a general sharp lower bound and an upper bound, the focus is on square grids, square prisms, and square toruses. For each of these Cartesian products the exact value of the vertex visibility number is obtained. We conclude the paper with some problems for future investigation.

1.1 Concepts and notation

Here we provide further necessary definitions.

Let $G = (V(G), E(G))$ be a graph. The order of G is denoted by $n(G)$, its maximum degree by $\Delta(G)$, and the (open) neighborhood of $x \in V(G)$ by $N_G(x)$. A vertex x of G is *simplicial* if the subgraph induced by $N_G(x)$ is complete. A vertex of G is *universal* if it is adjacent to all other vertices of G. A *double star* is a tree in which exactly two vertices are not leaves.

The distance function $d_G(\cdot,\cdot)$ is the standard shortest-path distance. The *eccentricity* $\mathrm{ecc}_G(x)$ of a vertex x is the maximum distance between x and the other vertices of G. A vertex z is an *eccentric vertex* of x if $d_G(x, z) = \mathrm{ecc}_G(x)$. The open interval $I(x, y)$ between vertices x and y is the set of all vertices that lie on shortest x, y-paths other than x and y. When every two vertices x and y of G are connected by a unique shortest x, y-path, G is called *geodetic*. A vertex y is *maximally distant* from x if $d_G(x, y) \geq d_G(x, z)$, for every $z \in N_G(y)$. (This notion was introduced in [18] as a tool to study the strong metric dimension.) The collection of all maximally distant vertices from x is denoted by $\mathrm{MD}_G(x)$.

If $S \subseteq V(G)$, then we say that $x, y \in V(G)$ are *S-visible*, if there exists a shortest x, y-path P such that $V(P) \cap S \subseteq \{x, y\}$. The set S is a *mutual-visibility set* if any two vertices from S are S-visible. The cardinality of a largest mutual-visibility set of G is the *mutual-visibility number* $\mu(G)$ of G. A mutual-visibility set of cardinality $\mu(G)$ is a *μ-set*. We also say that a vertex y is *S-visible from x* if there exists a shortest y, x-path P such that $V(P) \cap S \subseteq \{x, y\}$.

The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G) \times V(H)$, vertices (g,h) and (g',h') of $G \,\square\, H$ are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. A *G-layer* is a subgraph of $G \,\square\, H$ induced by $V(G) \times \{h\}$ for some $h \in V(H)$, denoted by G^h. Analogously, for $g \in V(G)$ we have the *H-layer* ${}^gH$.

Finally, for a positive integer k, the set $\{1, \ldots, k\}$ is denoted by $[k]$.

2 Shortest-path trees and computational complexity

To study the computational complexity of finding the x-visibility number $v_x(G)$ for a fixed vertex x of a graph G, we introduce the following decision problem.

Definition 2.1 x-VISIBILITY *problem:*
INSTANCE*: A graph G, a vertex $x \in V(G)$, a positive integer $k \leq n(G)$.*
QUESTION*: $v_x(G) \geq k$?*

In this section we prove that the x-VISIBILITY problem is NP-complete even for graphs of diameter 2, so that finding $\mathrm{vv}(G)$ is an NP-hard problem. This result is in sharp contrast with the computational complexity of the vertex position problem, which is solvable in $O(n^4 \log(n))$ time, as shown in [22].

The intrinsic difficulty of the x-VISIBILITY problem lies in the fact that the value $\mathrm{vv}(G)$ is equal to the largest possible number of leaves of a shortest-path tree of G. These trees are defined as follows. Let G be a graph and $x \in V(G)$. Then a tree rooted in x constructed by the BFS search is called a *shortest-path tree* of G. That is, it is a rooted spanning tree T, such that $d_T(x,y) = d_G(x,y)$ for every $y \in V(G)$.

Theorem 2.2 *If G is a connected graph, then $\mathrm{vv}(G)$ is equal to the largest possible number of leaves of a shortest-path tree of G.*

In view of Theorem 2.2, we first recall that the problem of finding a spanning tree with maximum number of leaves has been heavily researched and is computationally difficult. For instance, the problem is NP-hard as well as APX-hard for cubic graphs, see [2]. On the other hand, a 2-approximation algorithm is known for this problem [21]. We now demonstrate that the difference between the maximum number of leaves in a spanning tree of G and $\mathrm{vv}(G)$ can be arbitrarily large.

Let $\ell(G)$ denote the maximum number of leaves in a spanning tree of a graph G. Let G_n denote the graph obtained by taking n copies of the graph G in Figure 1 and identifying the vertex x. Then $\ell(G_n) = 11n$. By Lemma 3.2, it is straight forward to check that $v_x(G_n) = v_y(G_n) = v_z(G_n) = 10n$ while $v_a(G_n) = 9$, $v_b(G_n) = 8$ and $v_c(G_n) = 8$. Thus by the symmetry of the graph and by Lemma 1.1, we can conclude that $\mathrm{vv}(G_n) = 10n$. Hence the difference $\ell(G_n) - \mathrm{vv}(G_n)$ is n, which we summarize in the following result.

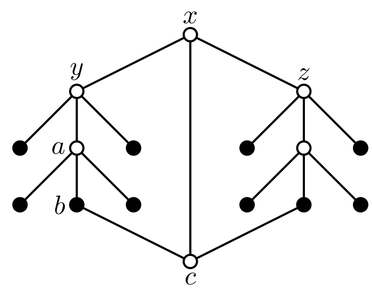

Figure 1: A graph G with $\ell(G) = 11$ but $\mathrm{vv}(G) = \mu_x(G) = 10$ (also attained at y and z).

Proposition 2.3 *Given a graph G, the difference between $\ell(G)$ and $vv(G)$ can be arbitrarily large.*

We are now ready to prove that the x-VISIBILITY problem is hard to solve.

Theorem 2.4 *The x-VISIBILITY problem is NP-complete even for graphs of diameter 2.*

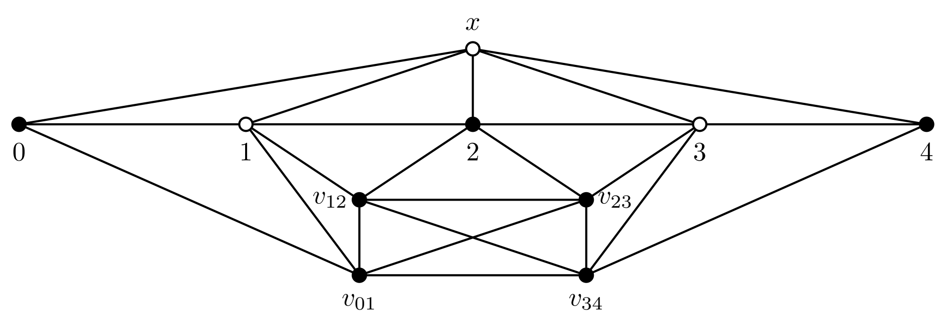

Figure 2: The graph G' for $G = P_5$.

The above Theorem 2.4 clearly implies that also finding $\mathrm{vv}(G)$ is an NP-hard problem.

3 General lower and upper bounds

In view of the hardness of the problem studied, as established in Section 2, in this section we prove general lower and upper bounds for the vertex visibility numbers.

Along the way, several exact values are also determined.

If S be a μ-set of G and $x \in S$, then $S \setminus \{x\}$ is an x-visibility set of G. Also, if $x \in V(G)$, then $N_G(x)$ is an x-visibility set. Thus we have the following general bounds:

$$\max\{\mu(G) - 1, \Delta(G)\} \le \mathrm{vv}(G) \le n(G) - 1. \tag{1}$$

The difference $\mathrm{vv}(G) - (\mu(G) - 1)$ can be arbitrarily large. For instance, it can be deduced from [6, Theorem 3.2.(i)] that if $n \ge 6$, then $\mu(K_2 \,\square\, C_n) = 6$, while on the other hand $\mathrm{vv}(K_2 \,\square\, C_n) \ge n$. The first assertion of the following result follows from (1), while the second is also straightforward to verify, hence we omit the proof.

Proposition 3.1 *If G is a connected graph with $n(G) \ge 2$, then the following properties hold.*

1. $\mathrm{vv}(G) = n(G) - 1$ *if and only if G has a universal vertex.*
2. $\mathrm{vv}(G) = n(G) - 2$ *if and only if G contains no universal vertex and contains a spanning double star.*

We now say that a vertex y is a *stress vertex* for x if there exists a maximally distant vertex z of x such that y lies on every x, z-shortest path in G. Note that, every cut vertex y, $y \neq x$, is a stress vertex for x. Let $\mathrm{str}_G(x)$ denote the number of stress vertices for x.

Lemma 3.2 *If x is a vertex of a connected graph G, then there exists a v_x-set S with the following properties:*

1. *every $y \in V(G) \setminus \{x\}$ is S-visible from x;*
2. $\mathrm{MD}_G(x) \subseteq S$*;*
3. *S contains no stress vertex for x.*

In addition, $|\,\mathrm{MD}_G(x)| \le v_x(G) \le n(G) - \mathrm{str}_G(x) - 1$.

Note that Lemma 3.2 implies that if G is a geodetic graph, then $v_x(G) = |\,\mathrm{MD}_G(x)|$ for every $x \in V(G)$. For the block graphs, which form a special case of geodetic graph, more specific result can be stated. In fact, this is an example where the lower and the upper bound in Lemma 3.2 coincide. Denoting by $s(G)$ the number of simplicial vertices of a graph G, the result for block graphs reads as follows.

Proposition 3.3 *If G is a block graph different from a complete graph then,*

$$\mathrm{vv}(G) = s(G).$$

We continue with the next upper bound depending on the order and the maximum degree.

Theorem 3.4 *If G has no universal vertex, then*

$$\mathrm{vv}(G) \leq \left\lfloor \frac{n(G)\Delta(G) - 1}{\Delta(G) + 1} \right\rfloor ,$$

and the bound is sharp.

We next bound the x-visibility number using the eccentricity of x.

Proposition 3.5 *If G is a graph and $x \in V(G)$, then*

$$\frac{n(G) - 1}{\mathrm{ecc}_G(x)} \leq v_x(G) \leq n(G) - \mathrm{ecc}_G(x) ,$$

and the bounds are sharp.

4 Cartesian products

In this section, we consider the Cartesian product graphs and first prove general lower and upper bounds on their vertex visibility number. Next, we focus on square grids ($P_n \,\Box\, P_n$), square prisms ($P_n \,\Box\, C_n$), and square toruses ($C_n \,\Box\, C_n$), and determine exact values for the vertex visibility number in all three cases. The general bounds read as follows.

Proposition 4.1 *If G and H are graphs with $n(G) \geq n(H)$, then*

$$\max\{\Delta(G)n(H), \Delta(H)n(G)\} \leq \mathrm{vv}(G \,\Box\, H) \leq (n(G) - 1)n(H) .$$

Both bounds are sharp, in particular, $\mathrm{vv}(K_m \,\Box\, K_n) = mn - m$ for $m \geq n \geq 2$.

Next, we will cover each of square grids, square prisms, and square toruses in their own subsections.

4.1 Square grids

In this subsection we determine the vertex visibility number of square grids.

Theorem 4.2 *If $n \geq 4$, then*

$$\mathrm{vv}(P_n \,\Box\, P_n) = \frac{n^2 + n - 2}{2} .$$

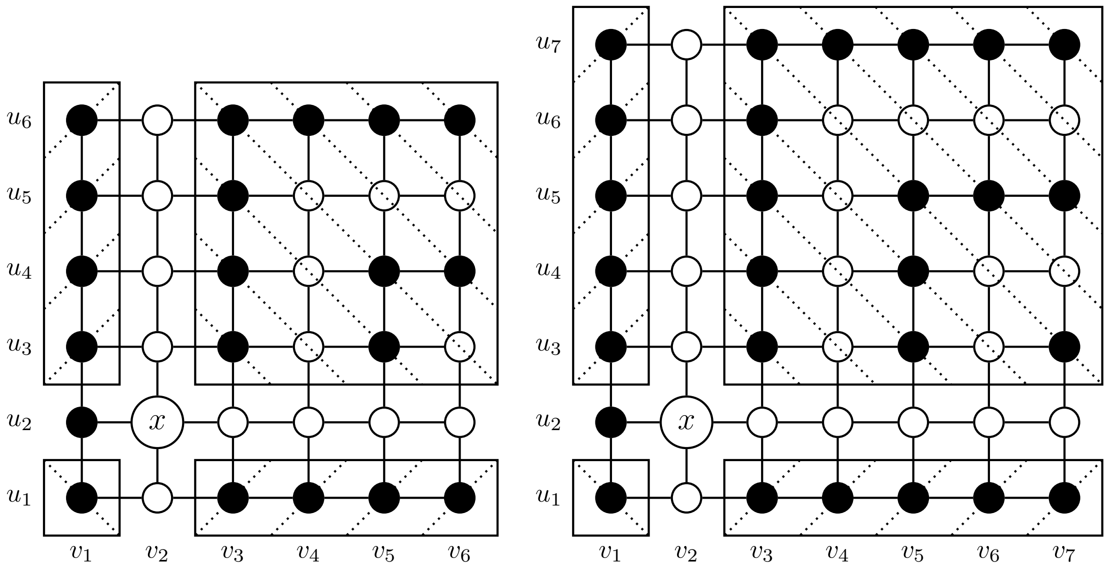

Figure 3: vv-sets of $P_6 \,\square\, P_6$ and $P_7 \,\square\, P_7$.

4.2 Square prisms

The proof technique used in Theorem 4.2 can be used to determine the vertex visibility number of square prisms.

Theorem 4.3 *If $n \geq 4$, then*

$$\mathrm{vv}(P_n \,\square\, C_n) = \begin{cases} \frac{n^2+3}{2}; & n \equiv 1 \bmod 4, \\ \frac{n^2+n-2}{2}; & n \equiv 3 \bmod 4, \\ \frac{2n^2+n}{4}; & n \equiv 0 \bmod 4, \\ \frac{2n^2+n-2}{4}; & n \equiv 2 \bmod 4. \end{cases}$$

4.3 Square toruses

Theorem 4.4 *If $n \geq 4$, then*

$$\mathrm{vv}(C_n \,\square\, C_n) = \begin{cases} \frac{n^2-1}{2}; & n \equiv 1 \bmod 4, \\ \frac{n^2+3}{2}; & n \equiv 3 \bmod 4, \\ \frac{n^2+2}{2}; & n \text{ even}. \end{cases}$$

The results of Theorems 4.2, 4.3 and 4.4 are summarized in Table 1.

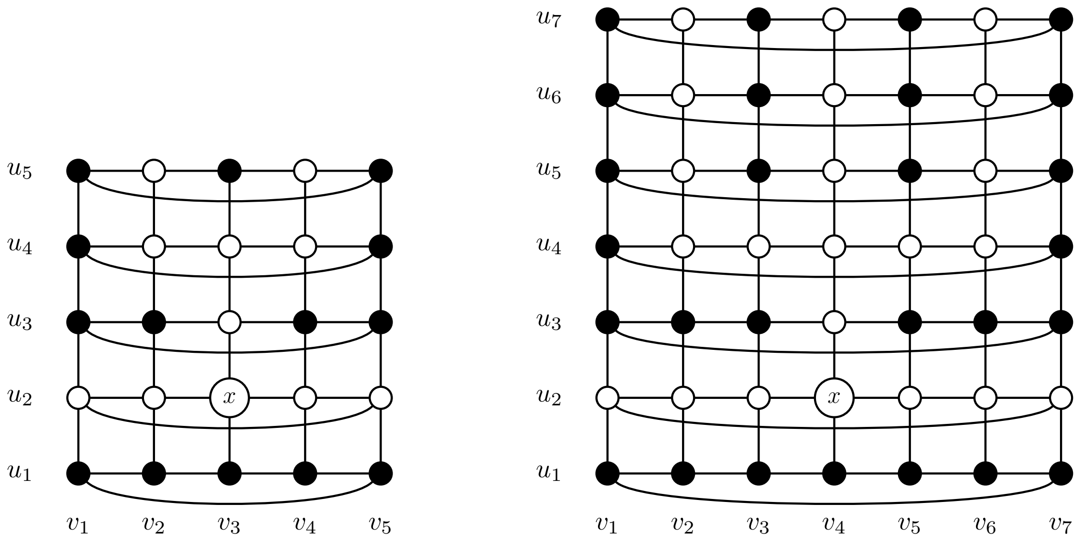

Figure 4: vv-sets of $P_5 \,\square\, C_5$ and $P_7 \,\square\, C_7$.

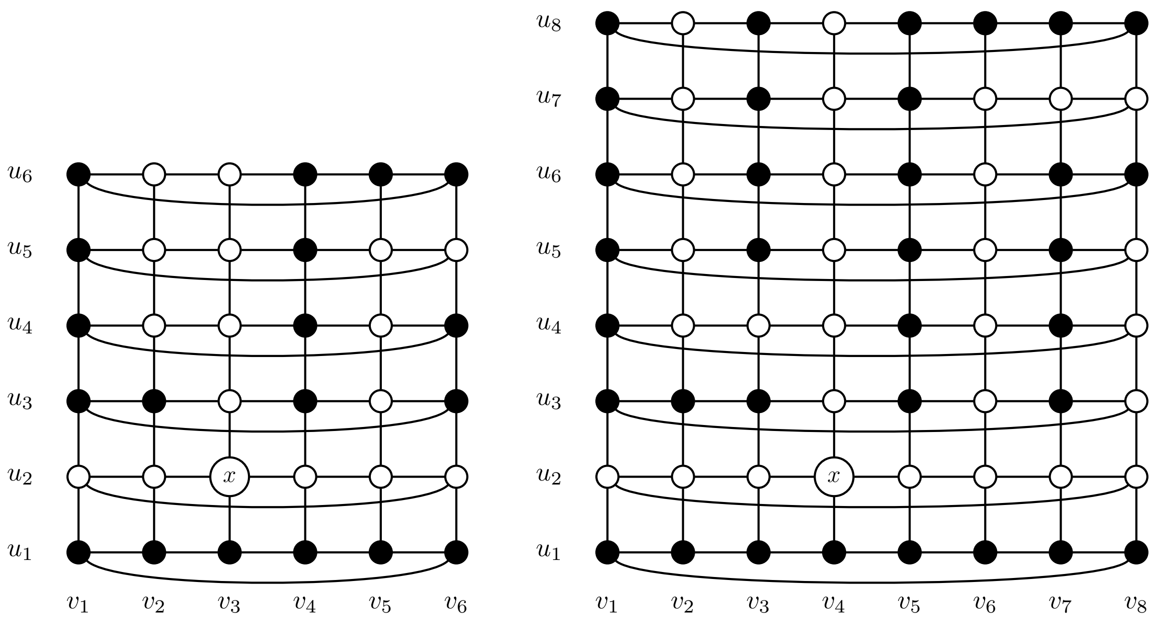

Figure 5: vv-sets of $P_6 \,\square\, C_6$ and $P_8 \,\square\, C_8$.

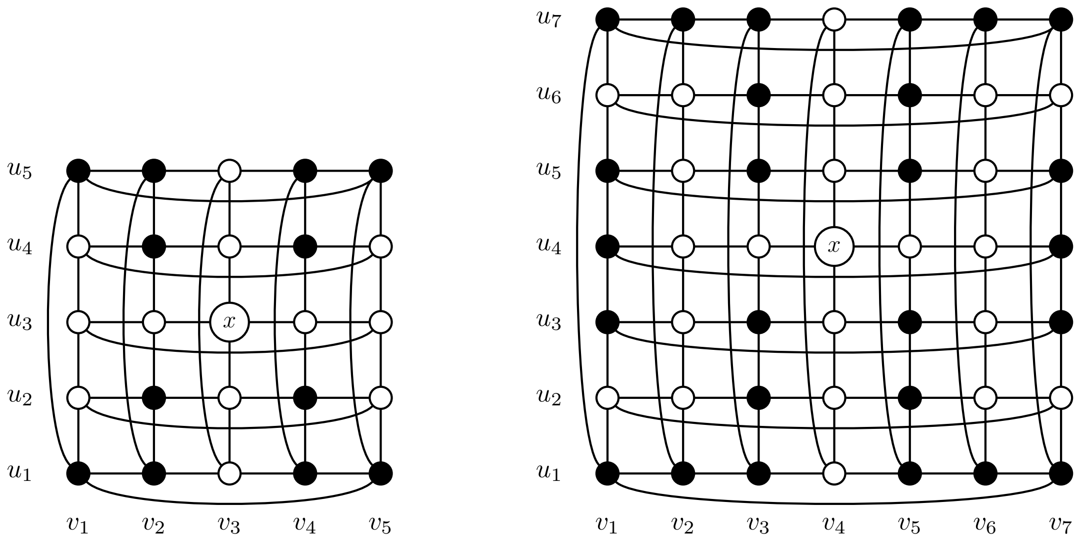

Figure 6: vv-sets of $C_5 \square C_5$ and $C_7 \square C_7$.

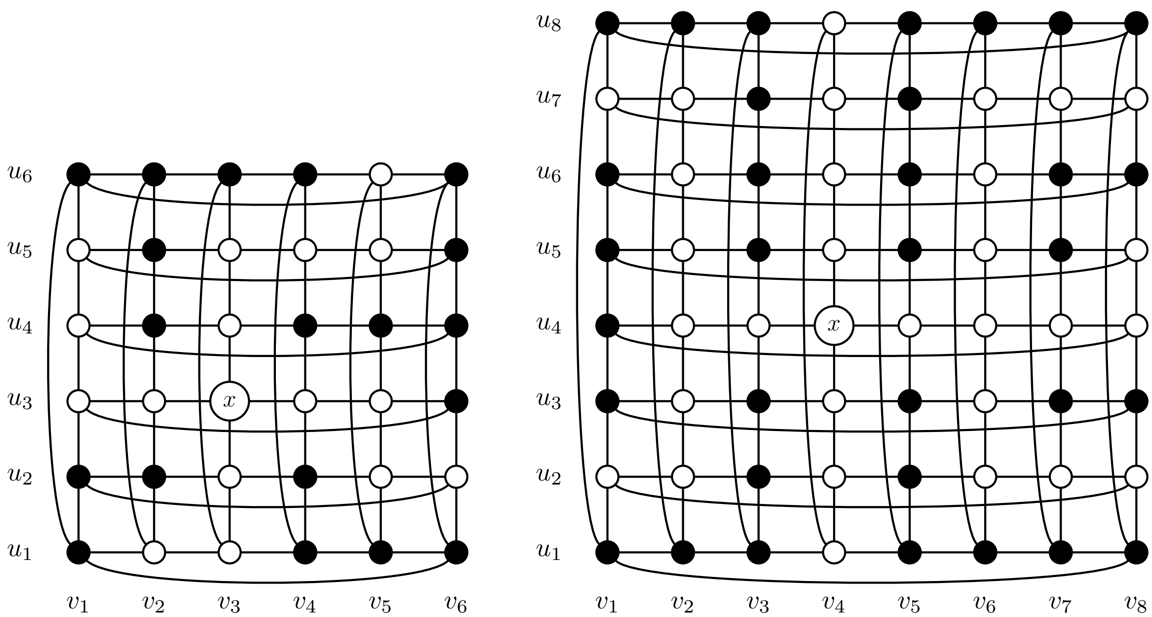

Figure 7: vv-sets of $C_6 \square C_6$ and $C_8 \square C_8$.

	$n \equiv 1 \bmod 4$	$n \equiv 3 \bmod 4$	$n \equiv 0 \bmod 4$	$n \equiv 2 \bmod 4$
$P_n \,\square\, P_n$	$\frac{n^2+n-2}{2}$	$\frac{n^2+n-2}{2}$	$\frac{n^2+n-2}{2}$	$\frac{n^2+n-2}{2}$
$P_n \,\square\, C_n$	$\frac{n^2+3}{2}$	$\frac{n^2+n-2}{2}$	$\frac{2n^2+n}{4}$	$\frac{2n^2+n-2}{4}$
$C_n \,\square\, C_n$	$\frac{n^2-1}{2}$	$\frac{n^2+3}{2}$	$\frac{n^2+2}{2}$	$\frac{n^2+2}{2}$

Table 1: The vertex visibility number of square grids, prisms, and toruses

5 Concluding remarks

We conclude the paper with some problems that deserve attention in the future.

Related to the sharp example of Proposition 4.1 we pose:

Problem 5.1 *Determine the vertex visibility number of the Cartesian product of an arbitrary product of finitely many complete graphs.*

Having studied the vertex visibility problem in the context of the Cartesian product, it is also worthwhile to explore it in other graph products. In particular, mutual-visibility has already been studied on strong products [8], hence we pose:

Problem 5.2 *Study the vertex visibility in the strong product of two graphs.*

The variety of mutual-visibility problems and the variety of the general position problems have been studied in [20] on Sierpiński graphs S_3^n, $n \geq 3$. Therefore, investigating the vertex visibility problem in these graphs may also be of interest.

Problem 5.3 *Determine the vertex visibility number of the Sierpiński graphs* S_3^n, $n \geq 3$.

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